NEW MODIFIED TWO-STEP JUNGCK ITERATIVE METHOD FOR SOLVING NONLINEAR FUNCTIONAL EQUATIONS

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ABSTRACT: *In this paper, we present a new modified two-step Jungck iterative method (NMJIM) for solving nonlinear functional equations and analyzed. The new modified two-step Jungck iterative method has convergence of order five and efficiency index 2.2361 which is larger than all Jungck type iterative methods and the methods discussed in Table 1. The new modified two-step Jungck iterative method converges faster than the methods discussed [1 − 19].*

The comparison tables demonstrate the faster convergence of new modified two-step Jungck method

Keywords-: fixed point method, Jungck iterative method, Nonlinear functional equation, Newton's method..

1. INTRODUCTION

The problem, to recall, is solving equations in one variable. We are given a function f and would like to find atleast one solution of the equation $f(x)=0$. Note that, we do not put any restrictions on the function f; we need to be able to evaluate the function; otherwise, we cannot even check that a given $x = \xi$ is true, that is $f(r)=0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior". The more we assume,the more potential we have, on the one hand, to develop fast iteration scheme for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This a fundamental paradigm in numerical analysis.

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method [4].

In the fixed-point iteration method for solving nonlinear equation $f(x)=0,$ the equation is usually rewritten as

$$x = g(x), \qquad (1.1)$$

where

(i) there exists $[a,b]$ such that $g(x)\in[a,b]$ for all $x\in[a,b]$,

(ii) there exists $[a,b]$ such that $|g(x)|\le L < 1$ for all $x\in[a,b]$.

Considering the following ite;ration scheme

$$x_{n+1} = g(x_n),\ n = 0,1,2..., \quad (1.2)$$

and starting with a suitable initial approximation x_0, we built up a sequence of approximations, say $\{x_n\}$, for the solution of nonlinear equation, say ξ . the scheme will be converge to ξ, provided that:

(i) the initial approximation x_0 is chosen in the interval $[a,b]$,

(ii) $|g'(x)| < 1$ for all $x\in[a,b]$,

(iii) $a\le g(x)\le b$ for all $x\in[a,b]$.

It is well known that the fixed point method has first order convergence.

2 Preliminaries

We define the basic concepts and relevant results required in this paper as follows.

Definition 2.1. [6] Let X be a Banach space and Y be an arbitrary set. Let $S,T : Y\to X$ and $T(Y)\subseteq S(Y)$.

For any $x_0\in Y,$ consider the following iterative scheme:

$$Sx_{n+1} = Tx_n ,\ n = 0,1,2...$$

This iteration scheme is called Jungck iteration scheme and was introduced in 1976.

Theorem 2.2. [6] Let f be a continuous mapping of a complete metric space (X,d) into itself. Then f has a fixed point in X iff there exists $a\in(0,1)$ and a mapping $g : X\to X$ which commutes with f and satisfies $g(X)\subset f(X)$ and

$$d(g(x),g(y))\le ad(f(x),f(y)) \qquad (2.1)$$

for all $x,y\in X$. Indeed, f and g have a unique common fixed point if (2.1) holds.

Definition 2.3. [2] Let (X,d) be a metric space and $T,S : X\to X$. Then x is called a Coincidence (common fixed) point of T and S respectively, if $x\in X$ such that $x = T(x) = S(x)$.

Theorem 2.4. [2] Let T be a multi-valued mapping from a metric space (X,d) to $CL(X)$ and $\Phi\in\Psi$. If there exists a mapping $f : X\to X$ such that $T(X)\subseteq f(X)$, and for each $x,y\in X$,

(1)

$$H\left(Tx,Ty\right)\le\Phi\left(\begin{matrix} max\{D\left(fx,Tx\right),D\left(fy,Ty\right) \\ ,D\left(fx,Ty\right),D\left(fy,Tx\right),d\left(fx,fy\right)\}\end{matrix}\right),$$

(2) $\Phi(t) < qt$ for each $t > 0$, for some fixed $0 < q < 1$,

(3) there exists an $x_0 \in X$ such that T is asymptotically regular at $x_0 \in X$,

(4) X is$(T, f, x_0)-$ orbitally complete, then T and f have a coincidence point.

Definition 2.5. [16] Let $\{x_n\}$ converges to α .if there exist an integer p and real positive constant C such that

$$\lim_{x\to\infty}\left|\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}\right|=C$$

then p is called the order and C is called the constant of convergence.

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of $g(x_n)$ given by

$$g\left(x_n\right)=g\left(x\right)+\frac{g'(x)}{1!}(x_n-x)+\frac{g''(x)}{2!}(x_n-x)^2$$

$$+,...,+\frac{g^{(k)}(x)}{k!}(x_n-x)^k+... \qquad (2.3)$$

Using (1.1) and (1.2) in (2.3) we have

$$x_{n+1}-x=g\left(x\right)+\frac{g'(x)}{1!}(x_n-x)+\frac{g''(x)}{2!}(x_n-x)^2$$

$$+,...,+\frac{g^{(k)}(x)}{k!}(x_n-x)^k+... \qquad (2.4)$$

and we can state the following result [4].

Theorem 2.6. [3] Suppose that $g\in C^n\left[a,b\right]$. if $g^k\left(x\right) = 0,$ for $k = 1,2,...,p-1$ and $g^k\left(x\right) \neq 0,$ the sequence $\{x_n\}$ is of order p.

During the last century, the numerical techniques for solving nonlinear equations have been successfully applied (see, e. g [1,3,5,7–15,17–19] and the references therein). T. J. McDougall et al. [19] modified the Newton's method and their modified Newton's method have convergence of order $\left(1+\sqrt{2}\right)$.

In this paper, we presented a new modified two-step Jungck iterative method (NMJIM) for solving nonlinear functional equations having convergence of order 5 and efficiency index 2.2361 extracted from [16] motivated by the technique of T. J. McDougall et al. [19]. The proposed new modified two-step Jungck iterative method is applied to solve some problems in order to assess its validity and accuracy.

3. MAIN RESULT

Consider the nonlinear equation

$$f\left(x\right) = 0,\ x\in\mathbb{R}. \qquad (3.1)$$

Let X be a Banach Space and Y be an arbitrary set. Let $S,T : Y\to X$; $T\left(Y\right)\subseteq S\left(Y\right)$, S is onto, S and T are differentiable. Suppose that ξ is simple zero of $f\left(x\right)$ and x_0 is an initial guess nearer to ξ . The equation (3.1) can be written as

$$Sx = Tx. \qquad (3.2)$$

Following the approach of [16], if $\frac{S'x}{T'x}\neq -1,$ we can modify (3.2) by multiplying $\theta\neq -1$ on both sides as follows

$$Sx + \theta Sx = \theta Sx + Tx$$

Implies that

$$Sx=\frac{\theta Sx+Tx}{\theta+1}=S_\theta x \text{ (say)} \qquad (3.3)$$

where θ is an arbitrary number. In order (3.3) to be efficient, we can choose θ such that $S'_\theta\ x = 0,$ which yields

$$\theta=-\frac{T'x}{S'x}, \qquad (3.4)$$

So that (3.3) takes the form

$$S_\theta(x)=\frac{-T'xSx+S'xTx}{S'x-T'x},\frac{T'x}{S'x}\neq -1 \qquad (3.5)$$

This is our modified Junck iterative method (MJIM).

This formation allows us to suggest the following iteration method for solving non-linear equation (3.1).

$$S_\theta(x_{n+1})=\frac{-T'x_nSx_n+S'x_nTx_n}{S'x_n-T'x_n},\frac{T'x_n}{S'x_n}\neq -1$$

Initially, we choose two starting points x and x^*. Then, we set $x^*=x,$ our modified two-step fixed point iterative method that we examine herein is given by

$$x_0^*=x_0,$$

$$Sx_1=\frac{-Sx_0T'\left(\frac{1}{2}[x_0+x_0^*]\right)+Tx_0S'\left(\frac{1}{2}[x_0+x_0^*]\right)}{S'\left(\frac{1}{2}[x_0+x_0^*]\right)-T'\left(\frac{1}{2}[x_0+x_0^*]\right)}, \qquad (3.6)$$

where,

$\frac{T'\left(\frac{1}{2}[x_0+x_0^*]\right)}{S'\left(\frac{1}{2}[x_0+x_0^*]\right)}\neq -1,$ which implies (for $n\geq 1$).

$$Sx_n^* = \frac{-Sx_0 T'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)+Tx_0 S'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)}{S'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)-T'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)}$$

$$S_\theta x_{n+1} = \frac{-Sx_0 T'\left(\frac{1}{2}[x_n+x_n^*]\right)+Tx_0 S'\left(\frac{1}{2}[x_n+x_n^*]\right)}{S'\left(\frac{1}{2}[x_n+x_n^*]\right)-T'\left(\frac{1}{2}[x_n+x_n^*]\right)}$$

the above equations labled as (3.7) and (3.8) respectively.

Where $\dfrac{T'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)}{S'\left(\frac{1}{2}[x_{n-1}+x_{n-1}^*]\right)} \neq -1, \dfrac{T'\left(\frac{1}{2}[x_n+x_n^*]\right)}{S'\left(\frac{1}{2}[x_n+x_n^*]\right)} \neq -1,$

These are the main steps of our new modified two-step Jungck iterative method.

The value of x_2 is calculated from x_1 using Sx_1 and the values of first derivatives of Sx and Tx evaluated at $\frac{1}{2}(x_1+x_1^*)$ (which is more appropriate value of the derivative to use than the one at x_1), and this same value of derivatives is re-used in the next predictor step to obtain x_3^*. This re-use of the derivative means that the evaluations of the starred values of x in (3.7) essentially come for free, which then enables the more appropriate value of the derivatives to be used in the corrector step (3.8)

4. CONVERGENCE ANALYSIS

Theorem 4.1. Let $f : X \subset R \to R$ for an open interval X and consider the nonlinear equation $f(x) = 0$ $(or\ Sx = Tx)$ has a simple root $\alpha \in X$, where $S, T : X \subset R \to R;$ S is onto, S and T are differentiable be sufficiently smooth in the neighborhood of the root α, then the order of convergence of new modified two-step Jungck iterative method given in (3.8) is at least 5.

Proof. To analysis the convergence of new modified two-step Jungck iterative method (3.8) let

$$S_\theta x = \frac{-SxT'\left(\frac{1}{2}[x+x^*]\right)+TxS'\left(\frac{1}{2}[x+x^*]\right)}{S'\left(\frac{1}{2}[x+x^*]\right)+T'\left(\frac{1}{2}[x+x^*]\right)}$$

Where, $\dfrac{T'\left(\frac{1}{2}[x+x^*]\right)}{S'\left(\frac{1}{2}[x+x^*]\right)} \neq -1,$

Let α be a simple zero of f $f(\alpha)=0$ $(\text{ or } S\alpha = T\alpha = \alpha)$, then we can easily deduce by using the software Maple that

$$\begin{aligned} &S_\theta \alpha = 0 \\ &S'_\theta \alpha = 0 \\ &S_\theta{}'' \alpha = 0 \\ &S_\theta{}''' \alpha = 0 \\ &S_\theta{}^{(iv)} \alpha = 0 \\ &S_\theta{}^{(v)} \alpha \neq 0. \end{aligned} \qquad (4.1)$$

We did not include the expression $S_\theta{}^{(v)}\alpha$ here, because it is very lengthy and consisted up to eleven pages. But, one can easily deduce that $S_\theta{}^{(v)}\alpha \neq 0$ by using Maple, MATLAB or Mathematica. Hence according to the theorem 2.6, new modified two-step Jungck iterative method (3.8) has at least fifth order convergence.

Weerakoon and Fernando [18], Homeier [7] and Frontini, and Sormani [13] have presented numerical methods having cubic convergence. In each iteration of these numerical methods three evaluations are required of either the function or its derivative. The best way of comparing these numerical methods is to express the of convergence per function or derivative evaluation, the so-called "efficiency" of the numerical method. On this basis, the Newton's method has an efficiency of $2^{\frac{1}{2}} \approx 1.4142$ the cubic convergence methods have an efficiency of $3^{\frac{1}{3}} \approx 1.4422$. Kuo [9] has developed several methods that each require two function evaluations and two derivative evaluations and these methods achieve an order of convergence of either five or six, so having efficiencies of $5^{\frac{1}{4}} \approx 1.4953$ and $6^{\frac{1}{4}} \approx 1.5651$ respectively. In these Kuo's methods the denominator is a linear combination of derivatives evaluated at different values of x, so that, when the starting value of x is not close to the root, this denominator may go to zero and the methods may not converge. Of the four 6th order methods suggested in Kuo [9], if the ratio of function's derivatives at the two value of x differ by a factor of more than three, then the method gives an infinite change in x. That is, the derivatives at the predictor and corrector stages can both be the same sign, but if their magnitudes differ by more than a factor of three, the method does not converge. Jarrat [14] developed a 4th order method that requires only one function evaluation and two derivative evaluations, and similar 4th order method have been described by Soleymani et al. [5]. Jarrat's method is similar to those of Kuo's methods in that if the ratio of derivatives at the predictor and corrector steps exceeds a factor of three, the method gives an infinite change in x. Jarratt's methods is similar to those of Kou in that if the ratio of the derivatives at the predictor and corrector steps exceeds a factor of three, the method gives an infinite change in x. Our new modified two-step

Jungck iterative method has efficiency $5^{\frac{1}{2}} \approx 2.2361$ larger than the efficiencies of all methods discussed above. The efficiencies of the methods, we have discussed are summarized in Table given

Table 1. Comparison of efficiencies of various methods

Method	Number of function or derivative evaluations	Efficiency index
Newton,quadratic	2	$2^{\frac{1}{2}} \approx 1.414$
Cubic methods	3	$3^{\frac{1}{3}} \approx 1.42$
Kou's 5th order	4	$5^{\frac{1}{4}} \approx 1.495$
Kou's 6th order	4	$6^{\frac{1}{4}} \approx 1.565$
Jarratt's 4th order	3	$4^{\frac{1}{3}} \approx 1.587$
Secant	1	$0.5(1+\sqrt{5}) \approx 1.6180$
NMJIM	2	$5^{\frac{1}{2}} \approx 2.236$

6. APPLICATIONS

In this section we included some nonlinear functions to illustrate the efficiency of our developed new modified two-step Jungck iterative method (NMJIM). We compare the NMJIM with Jungck iterative method (JIM) and modified Jungck iterative method (MJIM) as shown in Table 2.

Table 2. Comparison of JIM, MJIM and NMJIM

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
$f(x)=\ln(x)-\cos(x).Tx=\ln(x)$ and $Sx=\cos(x)$				
$x_0=1.5$				
JIM	187	187	$9.627806e-20$	
MJIM	4	8	$3.553544e-22$	1.302964
NMJIM	2	8	$3.009603e-22$	
$f(x)=\ln(x)+\sin(x).\ Tx=-\sin(x)$ and $Sx=\ln(x)$				
$x_0=0.5$				
JIM	59	59	$5.658891e-20$	
MJIM	4	8	$1.483752e-28$	0.5787136
NMJIM	2	4	$1.448443e-22$	
$f(x)=4\sin^{-1}(x)-4x\sqrt{(1-x^2)}+8x^2\cos^{-1}(x)-\pi.\quad Sx=4\sin^{-1}(x)$				
$Tx=4x\sqrt{(1-x^2)}-8x^2\cos^{-1}(x)+\pi,\quad x_0=0.52$				
JIM	194	194	$9.555657e-20$	
MJIM	5	10	$1.313012e-34$	0.5793642
NMJIM	2	4	$8.371034e-22$	
$f(x)=e^x-2\cos(e^x-2).\ Tx=\cos(e^x-2)$ and $Sx=e^x-2,$				
$x_0=1$				
JIM	104	104	$7.326364e-20$	
MJIM	4	8	$3.543049e-37$	1.0076239
NMJIM	2	4	$1.706417e-21$	
$f(x)=e^{x^2}-\frac{5}{e^{2x}}.Tx=e^{x^2}$ and $Sx=\frac{5}{e^{2x}}$				
$x_0=0.5$				
JIM	90	90	$8.688394e-20$	
MJIM	5	10	$1.102381e-31$	0.6153754
NMJIM	2	4	$3.234737e-21$	

Table 2. Shows the numerical comparisons of new modified two-step Jungck iterative method with Jungck iterative method and modified Jungck iterative method.

The columns represent the number of iterations N and the number of functions or derivatives evaluations N_f required to meet the stopping criteria, and the magnitude $\lvert f(x)\rvert$ of $f(x)$ at the final estimate x_n+1 .

7. CONCLUSIONS

A new MJIM for solving nonlinear functions has been established. We can concluded from tables (1, 2) that

1. The efficiency index of NMJIM is 2.2361 which is larger than the efficiency index of all Jungck type iterative method and the methods discussed in Table1.

2. The modified NMJIM has convergence of order five.

3. By using some examples the performance of NMJIM is also discussed. The NMJIM is performing very well in comparison to Jungck iterative method and modified Jungck as discussed in Table 2.

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